Journal of Approximation Theory **118,** 1–19 (2002)
doi:10.1006/jath.2002.3693

On the Completeness of the System $\{z^{\tau_n}\}$ in L^2

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Communicated by Manfred v. Golitschek
Received March 28, 2000; accepted April 1, 2002

Given an unbounded domain Ω located outside an angle domain with vertex at the origin, and a sequence of distinct complex numbers $\{\tau_n\}$ $(n = 1, 2, \ldots)$ satisfying $\frac{n}{|\tau_n|} \to D$ as $n \to \infty$ with $0 < D < \infty$, and $|\arg(\tau_n)| < \alpha < \frac{\pi}{2}$, we obtain a completeness theorem for the system $\{z^{\tau_n}\}$ $(n = 1, 2, \ldots)$ in $L^2_a[\Omega]$. The case with weight is also considered. © 2002 Elsevier Science (USA)

Key Words: mean square approximation; complex Müntz theorem; unbounded domain.

1. INTRODUCTION

Let Ω be a domain in the complex z plane, and $L^2_a[\Omega]$ be the set of all functions ϕ which are analytic in Ω with

$$\int\int_\Omega |\phi(z)|^2 \, d\sigma < \infty, \qquad z = x + iy,$$

where $d\sigma$ is the area element in the z plane (i.e., $d\sigma = dx\,dy = r\,dr\,d\theta$ for $z = x + iy = re^{i\theta}$). For $\phi, \psi \in L^2_a[\Omega]$, define the inner product

$$(\phi, \psi) = \int\int_\Omega \phi(z)\overline{\psi(z)} \, d\sigma,$$

and norm $||\phi|| = (\phi, \phi)^{1/2}$. With these, $L^2_a[\Omega]$ is a Hilbert space (see for example [6, Chap. 1]).

For $h_n \in L^2_a[\Omega]$ $(n = 1, 2, \ldots)$, we say that the system $\{h_n\}$ is complete in $L^2_a[\Omega]$ if for any $g \in L^2_a[\Omega]$,

$$\inf_{h \in S} ||g - h|| = 0,$$

where S denotes the linear span in $L^2_a[\Omega]$ of $\{h_n\}$.

[1]Research of this author is partially supported by a grant from NSERC of Canada.
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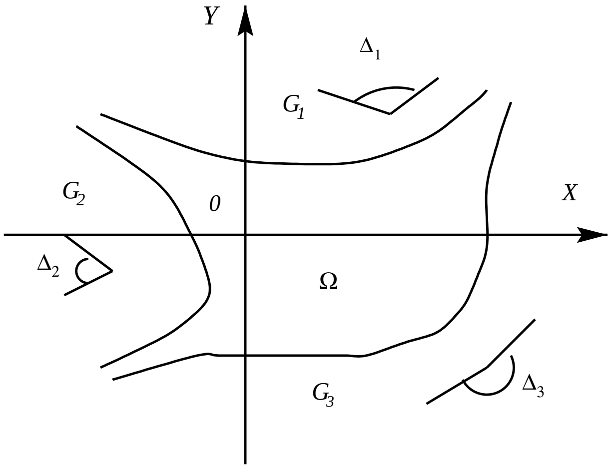

FIG. 1. Dzhrbasian domain.

It is natural to try to characterize, for example, those bounded domains Ω for which the polynomials are dense in $L_a^2[\Omega]$, that is to study the completeness of the system $\{1, z, z^2, \ldots\}$ in $L_a^2[\Omega]$. This is a very delicate topological and geometrical problem and a complete answer is known only in special cases. This is in sharp contrast with the simple topological characterization of domains obtained in the study of polynomial *uniform* approximation. We refer the reader to [6] for a nice survey of these questions (and many others).

In view of the Müntz–Szász theorem (see [9, Chap. 15] or [3, Chap. 6.2]), it is also natural to consider the completeness in $L_a^2[\Omega]$ of $\{z^{\tau_n}\}$ where $\{\tau_n\}$ is a sequence of complex numbers (and in general $0 \notin \Omega$). When Ω is bounded, this problem seems to have been first studied by Carleman [2].

These questions have also been considered on special unbounded domains by Dzhrbasian [4], Mergelyan [8] and Shen [10, 11]. We now proceed to describe their results.

Let Ω be an unbounded simply connected domain satisfying the following conditions (we will call such a domain a Dzhrbasian domain (see Fig. 1)):

Condition Ω(I). For $r > 0$, let $\sigma(r)$ denote the linear measure of the intersection of the circle $|z| = r$ and Ω. We suppose there exists $r_0 > 0$ such that for $r > r_0$,

$$\sigma(r) \leqslant e^{-p(r)},$$

where $p(r) > 0$ satisfies

$$p(r) = p(r_0) + \int_{r_0}^{r} \frac{\omega(t)}{t}\, dt \tag{1}$$

with $\omega(r) \geqslant 0$ and $\omega(r) \uparrow \infty$ as $r \to \infty$.

Condition Ω(II). It is assumed that the complement of Ω consists of m unbounded simply connected domains G_i $(i = 1, 2, \ldots, m)$, each containing an angle domain Δ_i with opening $\frac{\pi}{\alpha_i}$, $\alpha_i > \frac{1}{2}$.

Now let

$$\vartheta = \max\{\alpha_1, \ldots, \alpha_m\}, \tag{2}$$

where the α_j are the constants appearing in Condition Ω(II). Under the above conditions on Ω, Dzhrbasian proved that if

$$\int^{\infty} \frac{p(r)}{r^{1+\vartheta}}\, dr = +\infty$$

(where $\int^{\infty}$ means that the lower limit of the integral is sufficiently large), then the polynomial system $\{1, z, z^2, \cdots\}$ is complete in $L_a^2[\Omega]$.

Based on the results of Dzhrbasian, Shen [10] studied the completeness of the system $\{z^{\tau_n}\}$ in $L_a^2[\Omega]$, where $\{\tau_n\}_{n=1}^{\infty}$ is a sequence of complex numbers satisfying the following conditions:

$$\text{the } \tau_n \text{ are all distinct} \quad \text{and} \quad \lim_{n\to\infty} |\tau_n| = \infty, \tag{I}$$

$$\lim_{n\to\infty} \frac{n}{|\tau_n|} = D \qquad (0 < D < \infty), \tag{II}$$

$$\mathrm{Re}(\tau_n) > 0, \qquad |\mathrm{Im}(\tau_n)| < C, \tag{III}$$

where C is a constant.

Shen also assumed that Ω is a Dzhrbasian domain, but added the requirement that the vertex of Δ_1 is at the origin (hence that Ω does not contain 0 (see Fig. 2). Shen proved that if

$$2\alpha_1(1 - D) < 1,$$

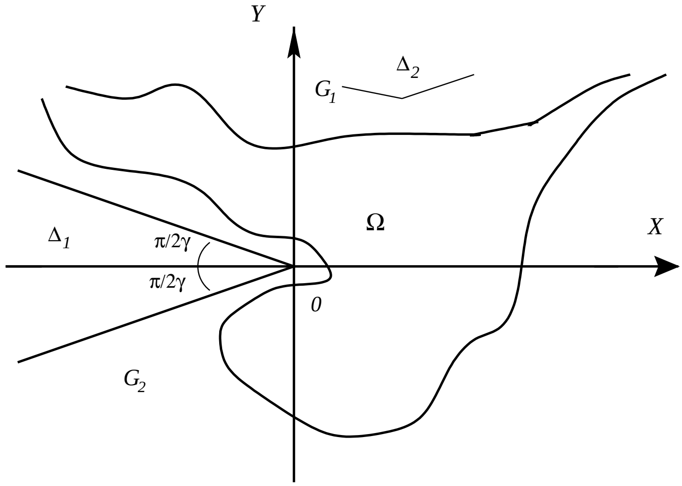

FIG. 2. Domain Ω.

and for some ε_0,

$$\int^{\infty} \frac{p(r)}{r^{1+\eta}}\,dr = \infty,$$

where

$$\eta = \max\left(\vartheta, \frac{1}{\frac{1}{\alpha_1} - 2(1-D)} + \varepsilon_0\right), \tag{3}$$

then the system $\{z^{\tau_n}\}_{n=1}^{\infty}$ is complete in $L_a^2[\Omega]$.

It follows from Condition (III) that $\arg(\tau_n) \to 0$ as $n \to \infty$. In this paper, we will relax this condition, allowing $\operatorname{Im}(\tau_n) \to \infty$ as $n \to \infty$. Our definition of η will reduce to Shen's definition (3) when the τ_n will lie in a strip. More precisely, we assume that $\{\tau_n\}$ satisfies Conditions (I) and (II), and the condition

$$|\arg(\tau_n)| < \alpha < \frac{\pi}{2} \tag{IIIa}$$

rather than (III).

We also assume that Ω is a Dzhrbasian domain, with the added requirement that Δ_1 is the angle domain

$$\Delta_1 = \left\{ z : |\arg(z) - \pi| < \frac{\pi}{2\gamma} \right\}, \tag{4}$$

where γ is a constant satisfying $\gamma > \frac{1}{2}$.

Some results of Shen obtained in [11], in particular, the residue estimate on the expansion of generalized Dirichlet polynomials (see Lemmas 2.1, 2.2 and 2.3 below), will play a very important role in this paper.

We end this section with some very elementary facts which we will need later.

LEMMA 1.1 (see, for example, Gaier [6, Chap. 1]). *A necessary and sufficient condition for the system $\{h_n\}$ to be complete in $L_a^2[\Omega]$ is that: for any $f \in L_a^2[\Omega]$, if $(f, h_n) = 0$ for all h_n, then $f(z) \equiv 0$.*

LEMMA 1.2. *Under our assumptions on τ_n and Ω, we have*

$$z^{\tau_n} \in L_a^2[\Omega], \qquad n = 1, 2, \ldots \, .$$

Remark 1. Of course, if τ_n is a non–negative integer, z^{τ_n} is an entire function. But in general, to define $z^{\tau_n} = \exp(\tau_n \log z)$, we need to fix a branch of the logarithm. For example, we can always choose the principal branch of $\log z$ (which we will denote by $\operatorname{Log} z$) since it is well defined on Ω (recall that Ω is located outside Δ_1 (see Fig. 2)).

Proof of Lemma 1.2. Let $z = re^{i\theta}, \tau_n = |\tau_n| e^{i\theta_n}$, then it is easy to see that

$$|z^{\tau_n}| = r^{|\tau_n| \cos \theta_n} e^{-|\tau_n| \theta \sin \theta_n}.$$

Since $|\theta_n| < \alpha < \frac{\pi}{2}$, and when $z \in \Omega$, $|\theta| < \pi - \pi/(2\gamma)$, it follows that there is a constant $c > 0$ such that for $z \in \Omega$,

$$|z^{\tau_n}| < (cr)^{|\tau_n|}.$$

Thus, by the assumptions on Ω, for fixed $n = 1, 2, \ldots$,

$$\begin{aligned} \int\int_\Omega |z^{\tau_n}|^2 \, dx \, dy &\leqslant \int_0^{r_0} 2\pi r (cr)^{2|\tau_n|} \, dr + \int_{r_0}^\infty \sigma(r)(cr)^{2|\tau_n|} \, dr \\ &\leqslant \frac{2\pi c^{2|\tau_n|} r_0^{2|\tau_n|+2}}{2|\tau_n| + 2} + c^{2|\tau_n|} \int_{r_0}^\infty e^{-p(r)} r^{2|\tau_n|} \, dr < \infty. \end{aligned}$$

Here we used Condition Ω(I). Indeed, since n is fixed and $\omega(r) \uparrow \infty$ as $r \to \infty$, we have $\omega(r) > 2|\tau_n| + 2$ for r sufficiently large, say $r > r_1$. Without loss of generality, we can assume that $r_1 > r_0$. Thus, by (1), we have for $r > r_1$,

$$p(r) > \int_{r_1}^r \frac{\omega(t)}{t} \, dt > (2|\tau_n| + 2) \int_{r_1}^r \frac{1}{t} \, dt$$

and

$$e^{-p(r)}r^{2|\tau_n|} < \exp\left[-(2|\tau_n|+2)\int_{r_1}^{r}\frac{dt}{t}\right]r^{2|\tau_n|} = \left(\frac{r}{r_1}\right)^{-(2|\tau_n|+2)} r^{2|\tau_n|} = \frac{r_1^{2|\tau_n|+2}}{r^2};$$

hence,

$$\int_{r_0}^{\infty} e^{-p(r)}r^{2|\tau_n|}\,dr < \infty. \quad \blacksquare$$

Remark 2. A similar argument shows that under our assumptions on Ω,

$$\int\int_{\Omega} d\sigma < \infty.$$

2. SOME LEMMAS

Consider the functions

$$T(z) = \prod_{k=1}^{\infty}\left(1 - \frac{z^2}{\tau_k^2}\right) \tag{5}$$

and

$$I(s) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-isy}}{T(iy)}\,dy, \qquad s = u + iv. \tag{6}$$

For sufficiently small $\delta > 0$, let

$$S_\delta = \{s = u + iv\colon\ |v| \leqslant \pi D\cos\alpha - \delta\pi\}. \tag{7}$$

Under Conditions (I), (II) and (IIIa), by [11, Sect. 1], we have

LEMMA 2.1. *Given* $\varepsilon > 0$,

$$\left|\frac{1}{T(iy)}\right| \leqslant C(\varepsilon)e^{(-\pi D\cos\alpha+\varepsilon)|y|},$$

where $C(\varepsilon)$ *is a constant which depends only on* ε.

LEMMA 2.2. *The integral in* (6) *is convergent uniformly and absolutely in* S_δ, *hence the function* $I(s)$ *is analytic and bounded in* S_δ *for any sufficiently small positive number* δ.

LEMMA 2.3. *There exists a sequence* $\{t_n\}$ *with* $n \geqslant t_n \geqslant (1-\lambda)n$ (λ *is a sufficiently small positive number*) *such that for* $s = u + iv \in S_\delta$,

$$\left| I(s) - \sum_{|\tau_k| < t_n} \frac{e^{-\tau_k s}}{T'(\tau_k)} \right| \leqslant \begin{cases} C^{t_n} e^{-ut_n \sin(\mu\pi)}, & \operatorname{Re}(s) = u \geqslant 0, \\ C^{t_n} e^{-ut_n}, & \operatorname{Re}(s) = u \leqslant 0, \end{cases} \tag{8}$$

where C *is a constant independent of* s *and* t_n, *while* μ *is a small positive number satisfying*

$$\tan(\mu\pi) < \frac{\delta}{D \sin \alpha}. \tag{9}$$

We will now transform the domain Ω of the z plane into a strip of the ξ plane:

Let $z = e^{\xi}$, $\xi = \xi_1 + i\xi_2$ (or equivalently, $\xi = \operatorname{Log} z$). Suppose that the image of Ω in the ξ plane is Ω'. By Condition $\Omega(II)$, since Ω is located outside the angle domain defined by (4), it is clear that Ω' must be located inside the strip

$$Q' = \left\{ \xi = \xi_1 + i\xi_2 \colon |\xi_2| < \pi\left(1 - \frac{1}{2\gamma}\right) \right\}.$$

Now we introduce two strips:

$$Q_\gamma = \left\{ s = u + iv \colon |v| < \pi D \cos \alpha - \pi\left(1 - \frac{1}{2\gamma}\right) \right\},$$

$$Q_\gamma^\delta = \left\{ s = u + iv \colon |v| \leqslant \pi D \cos \alpha - \delta\pi - \pi\left(1 - \frac{1}{2\gamma}\right) \right\}.$$

We assume now that

$$2\gamma(1 - D\cos\alpha) < 1,$$

thus $\pi D \cos\alpha - \pi(1 - \frac{1}{2\gamma}) > 0$, and we take δ so small that

$$0 < \delta < D\cos\alpha - 1 + \frac{1}{2\gamma}, \tag{10}$$

and thus,

$$\pi D \cos\alpha - \delta\pi - \pi\left(1 - \frac{1}{2\gamma}\right) > 0. \tag{11}$$

It is not hard to see then that if $s \in Q_\gamma^\delta$ and $\xi \in \Omega'$ (hence $\xi \in Q'$), we must have $|\operatorname{Im}(s - \xi)| < \pi D \cos\alpha - \delta\pi$, i.e., $s - \xi \in S_\delta$. Hence, for any $f(z) \in L_a^2[\Omega]$, we can define a function $G(s)$ for $s \in Q_\gamma^\delta$ by

$$G(s) = \int\int_{\Omega'} \overline{f(e^\xi)}|e^\xi|^2 I(s-\xi)\, d\xi_1\, d\xi_2, \qquad \xi = \xi_1 + i\xi_2. \tag{12}$$

By Lemma 2.2, when $\xi \in \Omega'$ is fixed, $I(s-\xi)$ hence $\overline{f(e^\xi)}|e^\xi|^2 I(s-\xi)$ is analytic for $s \in Q_\gamma^\delta$; when $s \in Q_\gamma^\delta$ is fixed, $\overline{f(e^\xi)}|e^\xi|^2 I(s-\xi)$ is measurable for $\xi \in \Omega'$; and $I(s-\xi)$ is bounded for $s \in Q_\gamma^\delta, \xi \in \Omega'$. Thus, it is not hard to prove using Remark 1.2 that $G(s)$ is analytic in Q_γ^δ (hence analytic in Q_γ since δ can be arbitrarily small) and bounded in Q_γ^δ (see, for example, [9, Chap. 10, Exercise 16; 1, Sect. 3]).

Now we prove an important lemma:

LEMMA 2.4. *If for* $s \in Q_\gamma^\delta$, $G(s) \equiv 0$ *where* $G(s)$ *is defined by* (12), *then*

$$\int\int_{\Omega} \overline{f(z)} z^n\, dz = 0, \qquad n = 0, 1, 2, \ldots\ . \tag{13}$$

Proof. Since $s - \xi \in S_\delta$ when $s \in Q_\gamma^\delta$, $\xi \in \Omega'$, it follows from Lemma 2.2 that for $s \in Q_\gamma^\delta$, the integral

$$I(s-\xi) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-i(s-\xi)y}}{T(iy)}\, dy$$

converges uniformly and absolutely with respect to $\xi = \xi_1 + i\xi_2 \in \Omega'$. Hence, we can interchange the order of integrations in (12):

$$G(s) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-isy}}{T(iy)}\left[\int\int_{\Omega'} \overline{f(e^\xi)}|e^\xi|^2 e^{iy\xi}\, d\xi_1 d\xi_2\right] dy \equiv 0, \qquad s \in Q_\gamma^\delta. \tag{14}$$

Let

$$l(y) = \frac{1}{T(iy)}\int\int_{\Omega'} \overline{f(e^\xi)}|e^\xi|^2 e^{iy\xi}\, d\xi_1\, d\xi_2.$$

It can be proved that $l(y) \in L^2(-\infty, \infty)$. Indeed, there exist $\varepsilon_1 > 0$, $C > 0$ such that for $y \in (-\infty, \infty)$,

$$\begin{aligned} |l(y)| &\leqslant \left|\int\int_{\Omega'} \overline{f(e^\xi)}|e^\xi|^2\, d\xi_1\, d\xi_2\right| \frac{1}{|T(iy)|} \max_{\xi \in \Omega'} |e^{iy\xi}| \\ &\leqslant C e^{(-\pi D \cos\alpha + \varepsilon)|y|} e^{\pi(1-\frac{1}{2\gamma})|y|} \\ &< C e^{-\varepsilon_1 |y|}. \end{aligned}$$

Here we used: (i) $f(z) \in L_a^2[\Omega]$; and (ii) Lemma 2.1 and the fact that $\pi D \cos\alpha - \pi(1 - 1/(2\gamma)) > 0$ to produce the constant $C > 0$ and $\varepsilon_1 > 0$, respectively.

Thus, by Plancherel theorem (see [9, Theorem 9.13]) and (14), we have

$$\int_{-\infty}^{\infty} |l(y)|^2 \, dy = 0,$$

and for $y \in (-\infty, \infty)$, $l(y) = 0$, i.e.,

$$\int\int_{\Omega'} \overline{f(e^{\xi})} |e^{\xi}|^2 e^{iy\xi} \, d\xi_1 \, d\xi_2 = 0, \tag{15}$$

since $l(y)$ is continuous in $(-\infty, \infty)$.

Consider the integral

$$H(w) = \int\int_{\Omega'} \overline{f(e^{\xi})} |e^{\xi}|^2 e^{w\xi} \, d\xi_1 \, d\xi_2 = \int\int_{\Omega} \overline{f(z)} z^w \, dx \, dy.$$

We claim that $H(w)$ is analytic in $\mathrm{Re}(w) > -\frac{1}{2}$. Indeed, for any $R > 0$, when $-\frac{1}{2} \leqslant \mathrm{Re}(w) \leqslant R$, $|\mathrm{Im}(w)| \leqslant R$,

$$\begin{aligned}
&\int\int_{\Omega} |\overline{f(z)} z^w| \, dx \, dy \\
&\quad = \int\int_{\Omega'} |\overline{f(e^{\xi})}| |e^{\xi}|^2 |e^{w\xi}| \, d\xi_1 \, d\xi_2 \\
&\quad = \int\int_{\Omega'} |\overline{f(e^{\xi})}| |e^{\xi}|^2 e^{\mathrm{Re}(w)\xi_1 - \mathrm{Im}(w)\xi_2} \, d\xi_1 \, d\xi_2 \\
&\quad \leqslant e^{R\pi(1-1/2\gamma)} \int\int_{\Omega'} |\overline{f(e^{\xi})}| |e^{\xi}|^2 |e^{\xi}|^{\mathrm{Re}(w)} \, d\xi_1 \, d\xi_2 \\
&\quad \leqslant e^{R\pi(1-1/2\gamma)} \left[\int\int_{\substack{\Omega' \\ |e^{\xi}| \geqslant 1}} |\overline{f(e^{\xi})}| |e^{\xi}|^2 |e^{\xi}|^R \, d\xi_1 \, d\xi_2 \right. \\
&\qquad \left. + \int\int_{\substack{\Omega' \\ |e^{\xi}| < 1}} |\overline{f(e^{\xi})}| |e^{\xi}|^2 |e^{\xi}|^{-1/2} \, d\xi_1 \, d\xi_2 \right] \\
&\quad \leqslant e^{R\pi(1-1/2\gamma)} \int\int_{\Omega'} |\overline{f(e^{\xi})}| |e^{\xi}|^2 [|e^{\xi}|^R + |e^{\xi}|^{-1/2}] \, d\xi_1 \, d\xi_2 \\
&\quad = e^{R\pi(1-1/2\gamma)} \int\int_{\Omega} |\overline{f(z)}| [|z|^R + |z|^{-1/2}] \, dx \, dy \\
&\quad \leqslant e^{R\pi(1-1/2\gamma)} \left[\int\int_{\Omega} |\overline{f(z)}|^2 \, dx \, dy \right]^{1/2} \left[\int\int_{\Omega} [|z|^R + |z|^{-1/2}]^2 \, dx \, dy \right]^{1/2}.
\end{aligned}$$

By using the same argument as in the proof of Lemma 1.2, we can get

$$\int\int_{\Omega} |z|^{2R}\, dx\, dy < \infty,$$

$$\int\int_{\Omega} |z|^{R-1/2}\, dx\, dy < \infty,$$

$$\int\int_{\Omega} |z|^{-1}\, dx\, dy < \infty.$$

Thus, for $-\frac{1}{2} \leqslant \mathrm{Re}(w) \leqslant R,\ |\mathrm{Im}(w)| \leqslant R,$

$$\int\int_{\Omega} |\overline{f(z)} z^{w}|\, dx\, dy < \infty,$$

and by using similar arguments to that in [1, Sect. 3], we can prove, using Remark 1.2, that $H(w)$ is analytic in $-\frac{1}{2} < \mathrm{Re}(w) < R,\ |\mathrm{Im}(w)| < R$. Since R can be arbitrarily large, therefore $H(w)$ is analytic in $\mathrm{Re}(w) > -\frac{1}{2}$.

By (15), for $y \in (-\infty, \infty), H(iy) = 0$. This implies that $H(w) = 0$ for $\mathrm{Re}(w) > -\frac{1}{2}$. In particular, $H(n) = 0, n = 0, 1, 2, \ldots$, i.e.,

$$\int\int_{\Omega} \overline{f(z)} z^{n}\, dx\, dy = 0, \qquad n = 0, 1, 2, \ldots . \quad \blacksquare$$

In order to prove our main theorem in Section 3, we need also the following two results:

LEMMA 2.5 (Carleman's Theorem (see Levin [7, p. 105])). *If* $g(w)$ *is analytic and bounded in the half-plane* $\mathrm{Im}(w) \geqslant 0$, *and*

$$\int_{-\infty}^{\infty} \frac{\log^{-}|g(t)|}{1+t^2}\, dt = \infty,$$

then $g(w) \equiv 0$.

LEMMA 2.6 (M.M. Dzhrbasian (see Mergelyan [8, Sect. 10, Lemma 1])). *Let* $p(r)$ *be given as in Condition* $\Omega(\mathrm{I})$, *let*

$$M_n = \int_{r_0}^{\infty} \exp[-p(r)] r^{n}\, dr$$

and

$$\Phi(r) = \sup_{n \geqslant 1} \frac{r^n}{\sqrt{M_{2n}}}.$$

Then there exists $q_0 > 0$ *such that for* r *sufficiently large,*

$$\log \Phi(r) \geqslant q_0 p(r).$$

3. A COMPLETENESS THEOREM

Now we give the main result of this paper:

THEOREM 1. *Assume that the domain* Ω *and the sequence* $\{\tau_n\}$ *satisfy Conditions* Ω(I), Ω(II), (I), (II), (IIIa) *and* (4) *given in Section 1. Moreover, assume*

$$2\gamma(1 - D\cos\alpha) < 1. \tag{16}$$

Let

$$\eta = \max\left\{\vartheta, \frac{1}{h} + \varepsilon_0\right\}, \tag{17}$$

where ϑ *is defined in* (2), ε_0 *is some positive number, and*

$$h = \max_{0<x<b}\left\{\frac{x}{\sqrt{D^2\sin^2\alpha + x^2}}\left[2D\cos\alpha - 2 + \frac{1}{\gamma} - 2x\right]\right\} \tag{18}$$

with $b = D\cos\alpha - 1 + \frac{1}{2\gamma}$.

If

$$\int^{\infty} \frac{p(r)}{r^{1+\eta}}\, dr = +\infty, \tag{19}$$

then the system $\{z^{\tau_1}, z^{\tau_2}, z^{\tau_3}, \ldots\}$ *is complete in* $L_a^2[\Omega]$.

Remark 3. Letting $\alpha \to 0$ in (18), we recover Shen's original condition (see (3)).

Proof of Theorem 1. By Lemma 1.1, we only need to prove that if $f \in L_a^2[\Omega]$, and

$$(f(z), z^{\tau_n}) = 0, \qquad n = 1, 2, 3, \ldots,$$

then $f(z) \equiv 0$. So, we assume that $(f(z), z^{\tau_n}) = 0, = 1, 2, 3, \ldots$.

Recalling the definition of $G(s)$ (see (12)), we first note that we only need to prove that $G(s) \equiv 0$ for $s \in Q_\gamma^\delta$. Indeed if so, by Lemma 2.4, we will have

$$(f(z), z^n) = 0, \qquad n = 0, 1, 2, \ldots, \tag{20}$$

and using M. Dzhrbasian's result, assuming that

$$\int^\infty \frac{p(r)}{r^{1+\omega}}\, dr = +\infty, \tag{21}$$

the system $\{z^n\}$ $(n = 0, 1, 2, \ldots)$ is complete in $L_a^2[\Omega]$. It follows easily from the conditions of our theorem that (21) is satisfied. Thus, by Lemma 1.1 and (20), we will have $f(z) \equiv 0$ for $z \in \Omega$.

Now we prove that for $s \in Q_\gamma^\delta$, $G(s) \equiv 0$. We will use Lemma 2.3. Let t_n be defined as in Lemma 2.3, i.e., $n \geqslant t_n \geqslant (1 - \lambda)n$ (where λ is a sufficiently small positive number).

By (12), for $s \in Q_\gamma^\delta$,

$$\begin{aligned} G(s) = {} & \int\int_{\Omega'} \overline{f(e^\xi)} |e^\xi|^2 \left[I(s - \xi) - \sum_{|\tau_k| < t_n} \frac{e^{-\tau_k(s-\xi)}}{T'(\tau_k)} \right] d\xi_1\, d\xi_2 \\ & + \int\int_{\Omega'} \overline{f(e^\xi)} |e^\xi|^2 \sum_{|\tau_k| < t_n} \frac{e^{-\tau_k(s-\xi)}}{T'(\tau_k)}\, d\xi_1\, d\xi_2 \\ =: {} & G_{1,t_n}(s) + G_{2,t_n}(s). \end{aligned}$$

Since $(f(z), z^{\tau_n}) = 0$, $n = 1, 2, 3, \ldots$, we have

$$\int\int_{\Omega'} \overline{f(e^\xi)} |e^\xi|^2 e^{\tau_n \xi}\, d\xi_1\, d\xi_2 = 0, \qquad n = 1, 2, 3, \ldots .$$

Thus,

$$G_{2,t_n}(s) = \sum_{|\tau_k| < t_n} \frac{e^{-\tau_k s}}{T'(\tau_k)} \int\int_{\Omega'} \overline{f(e^\xi)} |e^\xi|^2 e^{\tau_k \xi}\, d\xi_1\, d\xi_2 = 0.$$

Hence, for $s \in Q_\gamma^\delta$, $G(s) = G_{1,t_n}(s)$.

By Lemma 2.3, for $s = u + iv \in Q_\gamma^\delta$,

$$
\begin{aligned}
|G(s)| &= |G_{1,t_n}(s)| \\
&\leqslant C^{t_n}\Bigg[e^{-ut_n \sin(\mu\pi)} \int\int_{\substack{\Omega' \\ \mathrm{Re}(s-\xi)\geqslant 0}} |\overline{f(e^\xi)}||e^\xi|^2|e^\xi|^{t_n \sin(\mu\pi)}\, d\xi_1\, d\xi_2 \\
&\quad + e^{-ut_n} \int\int_{\substack{\Omega' \\ \mathrm{Re}(s-\xi)\leqslant 0}} |\overline{f(e^\xi)}||e^\xi|^2|e^\xi|^{t_n}\, d\xi_1\, d\xi_2\Bigg],
\end{aligned}
$$

where C (and C_1, C_2 below) are constants independent of n and s.

Hence, for $\mathrm{Re}(s) = u > 0$, $s \in Q_\gamma^\delta$,

$$
\begin{aligned}
|G(s)| &\leqslant C^{t_n}\left[\frac{\int\int_\Omega |f(z)||z|^{t_n}\, dx\, dy}{|e^s|^{t_n \sin(\mu\pi)}} + \frac{\int\int_\Omega |f(z)||z|^{t_n}\, dx\, dy}{|e^s|^{t_n}}\right] \\
&\leqslant C_1^{t_n} \frac{\int\int_\Omega |f(z)||z|^{t_n}\, dx\, dy}{|e^s|^{t_n \sin(\mu\pi)}}.
\end{aligned}
$$

By Schwarz's inequality,

$$
|G(s)| \leqslant C_1^{t_n} \frac{[\int\int_\Omega |f(z)|^2\, dx\, dy]^{1/2}[\int\int_\Omega |z|^{2t_n}\, dx\, dy]^{1/2}}{|e^s|^{t_n \sin(\mu\pi)}}.
$$

Note that

$$
\begin{aligned}
\int\int_\Omega |z|^{2t_n}\, dx\, dy &\leqslant \int_0^{r_0} 2\pi r r^{2t_n}\, dr + \int_{r_0}^\infty r^{2t_n}\sigma(r)\, dr \\
&\leqslant \frac{2\pi}{2t_n + 2} r_0^{2t_n+2} + \int_{r_0}^\infty r^{2t_n} e^{-p(r)}\, dr \\
&\leqslant c_1^{t_n} + c_2^{t_n} \int_{r_0}^\infty r^{2t_n} e^{-p(r)}\, dr \\
&\leqslant c_3^{t_n} \int_{r_0}^\infty r^{2t_n} e^{-p(r)}\, dr,
\end{aligned}
$$

where c_1–c_3 are positive constants independent of n and s.

Thus, since $f \in L_a^2[\Omega]$, we have

$$
|G(s)| \leqslant C_2^{t_n} \frac{[\int_{r_0}^\infty r^{2t_n} e^{-p(r)}\, dr]^{1/2}}{|e^s|^{t_n \sin(\mu\pi)}} \leqslant C_2^n \frac{[\int_{r_0}^\infty r^{2n} e^{-p(r)}\, dr]^{1/2}}{|e^s|^{(1-\lambda)n \sin(\mu\pi)}},
$$

where in the last step, we used the conditions $\mathrm{Re}(s) = u > 0$ and $n \geqslant t_n \geqslant (1 - \lambda)n$.

Since the above inequality holds for all $n = 1, 2, 3, \ldots,$ we have for $\mathrm{Re}(s) = u > 0,\ s \in Q_\gamma^\delta$,

$$|G(s)| \leqslant \inf_{n \geqslant 1} C_2^n \frac{[\int_{r_0}^{\infty} r^{2n} e^{-p(r)}\, dr]^{1/2}}{|e^s|^{(1-\lambda)n \sin(\mu\pi)}}.$$

Now using Lemma 2.6, for

$$M_n = \int_{r_0}^{\infty} r^n e^{-p(r)}\, dr$$

and

$$\Phi(\bar{r}) = \sup_{n \geqslant 1} \frac{\bar{r}^n}{\sqrt{M_{2n}}},$$

where $\bar{r} = c|e^s|^{(1-\lambda)\sin(\mu\pi)}$ with c a constant independent of s and n, then there exists a constant $q > 0$ such that for Re(s) sufficiently large (i.e., $\bar{r}$ sufficiently large),

$$\frac{1}{\Phi(\bar{r})} \leqslant e^{-q p(\bar{r})}.$$

Hence, for $s \in Q_\gamma^\delta$ and $\mathrm{Re}(s) > 0$ sufficiently large, we have

$$|G(s)| \leqslant \frac{1}{\Phi(\bar{r})} \leqslant e^{-q p(c|e^s|^{(1-\lambda)\sin(\mu\pi)})}. \tag{22}$$

We know that $G(s)$ is analytic in Q_γ and bounded in Q_γ^δ. In order to use Lemma 2.5 (Carleman's theorem), we transform Q_γ^δ (with respect to s) into the upper half-plane $\mathrm{Im}(w) \geqslant 0$:

(i) First, let $w_1 = e^s$. Then Q_γ^δ is transformed into an angle domain $|\arg(w_1)| \leqslant \pi l$, where by (11),

$$l = D \cos \alpha - \delta - 1 + \frac{1}{2\gamma} > 0. \tag{23}$$

(ii) Let $w_2 = w_1^{1/(2l)}$. The above angle domain is then transformed into the right half-plane $\mathrm{Re}(w_2) \geqslant 0$.

(iii) Finally, let $w = iw_2$. The right half-plane is then transformed into the upper half-plane $\mathrm{Im}(w) \geqslant 0$.

Now, we have

$$|e^s| = |w_1| = |w_2^{2l}| = |(-iw)^{2l}| = |w^{2l}|$$

and

$$G(s) = G(\log w_1) = G(\log w_2^{2l}) = G(\log(-iw)^{2l}).$$

Let $g(w) = G(\log(-iw)^{2l})$. Clearly, $g(w)$ is analytic and bounded in the half-plane $\mathrm{Im}(w) \geqslant 0$. By (22), for $\mathrm{Im}(w) \geqslant 0$ and $|w|$ sufficiently large, we have

$$|g(w)| \leqslant e^{-qp(c|w|^{2l(1-\lambda)\sin(\mu\pi)})} = e^{-qp(c|w|^m)}, \tag{24}$$

where $c, q > 0$ are constants independent of w, l is given by (23), and

$$m = 2l(1-\lambda)\sin(\mu\pi) = \sin(\mu\pi)\left[2D\cos\alpha - 2 + \frac{1}{\gamma} - 2\delta\right](1-\lambda). \tag{25}$$

Recall that in (25): (i) D, α and γ are fixed since they are determined by $\{\tau_n\}$ and Ω, respectively; (ii) $\lambda > 0$ can be taken arbitrarily small; (iii) $0 < \delta < D\cos\alpha - 1 + 1/(2\gamma)$ (see (16) and (10)); and (iv) $\mu > 0$ and $\tan(\mu\pi) < \delta/(D\sin\alpha)$ (see Lemma 2.3).

Let $\tan(\mu\pi) \to \delta/(D\sin\alpha)$, then

$$\sin(\mu\pi) \to \frac{\delta}{\sqrt{D^2\sin^2\alpha + \delta^2}},$$

and denote

$$m' = \frac{\delta}{\sqrt{D^2\sin^2\alpha + \delta^2}}\left[2D\cos\alpha - 2 + \frac{1}{\gamma} - 2\delta\right](1-\lambda). \tag{26}$$

It is clear that by (24), for $\mathrm{Im}(w) \geqslant 0$ and $|w|$ sufficiently large, we have

$$|g(w)| \leqslant e^{-qp(c|w|^{m'})}. \tag{27}$$

Since δ can be any number satisfying $0 < \delta < b = D\cos\alpha - 1 + 1/(2\gamma)$, letting

$$h' = \max_{0<\delta<b} m', \tag{28}$$

we see that in (27), if m' is replaced by h', the inequality still holds, i.e., for $\mathrm{Im}(w) \geqslant 0$ and $|w|$ sufficiently large, we have

$$|g(w)| \leqslant e^{-qp(c|w|^{h'})}. \tag{29}$$

We note that $h' = h(1-\lambda)$. Since $\lambda > 0$ can be taken arbitrarily small, we can choose λ sufficiently small such that

$$\frac{1}{h'} < \frac{1}{h} + \varepsilon_0,$$

where $\varepsilon_0 > 0$ is from (17). By Conditions (17)–(19), we have

$$\int^{\infty} \frac{p(r)}{r^{1+1/h'}}\, dr = +\infty. \tag{30}$$

Now we prove that $g(w) \equiv 0$ on $\mathrm{Im}(w) \geqslant 0$. This in turn will imply that $G(s) \equiv 0$ on Q_{λ}^{δ}, and thus the proof will be completed. Indeed, by (29),

$$\begin{aligned}\int^{\infty} \frac{\log|g(t)|}{t^2}\, dt &\leqslant \int^{\infty} \frac{-q\, p(ct^{h'})}{t^2}\, dt \\ &= -q \int^{\infty} \frac{p(r)}{(\frac{r}{c})^{2/h'}} \frac{1}{h'} \left(\frac{r}{c}\right)^{\frac{1}{h'}-1} \left(\frac{1}{c}\right) dr \\ &= -C \int^{\infty} \frac{p(r)}{r^{1+1/h'}}\, dr,\end{aligned}$$

where C is a positive constant.

Thus, by (30),

$$\int^{\infty} \frac{\log|g(t)|}{t^2}\, dt = -\infty;$$

hence,

$$\int^{\infty} \frac{\log|g(t)|}{1+t^2}\, dt = -\infty.$$

Similarly, we can get

$$\int_{-\infty} \frac{\log|g(t)|}{t^2}\, dt < \int_{-\infty} \frac{-q\, p(c|t|^{h'})}{t^2}\, dt = \int^{\infty} \frac{-q\, p(ct^{h'})}{t^2}\, dt = -\infty,$$

where $\int_{-\infty}$ means that the upper limit of the integral is a negative number with sufficiently large magnitude. Hence

$$\int_{-\infty} \frac{\log|g(t)|}{1+t^2}\, dt = -\infty.$$

Thus, we have

$$\int_{-\infty}^{\infty} \frac{\log|g(t)|}{1+t^2}\,dt = -\infty,$$

and by Lemma 2.5 (Carleman's theorem), $g(w) \equiv 0$. ∎

Remark 4. In (18), h is well defined. Indeed, let

$$A = 2b = 2D\cos\alpha - 2 + 1/\gamma,$$

$$B = D^2 \sin^2 \alpha$$

and

$$y(x) = \frac{x}{\sqrt{D^2\sin^2\alpha + x^2}}[2D\cos\alpha - 2 + 1/\gamma - 2x] = \frac{x}{\sqrt{B+x^2}}(A-2x).$$

Since $2\gamma(1 - D\cos\alpha) < 1$, $A > 0$, it is easy to verify that $y'(0) = A/\sqrt{B} > 0$, and $y'(A/2) = -(2A)/\sqrt{A^2+4B} < 0$. Hence, there exists an $\bar{x} \in (0, A/2) = (0, b)$ such that $y(\bar{x}) = h$.

4. CASE WITH WEIGHT

Assume that $p_0(z)$ is a real-valued function satisfying $p_0(z) \geqslant p(|z|) = p(r)$ for $|z| = r$ sufficiently large (say $r \geqslant r_0$), where $p(r)$ is defined by (1). In this section, we consider the completeness of $\{z^{\tau_1}, z^{\tau_2}, \ldots\}$ in $L_a^2[\Omega]$ with the weight $e^{-p_0(z)}$.

We say $f(z) \in L_a^2[\Omega, p_0]$, if $f(z)$ is analytic in Ω and

$$\int\int_{\Omega} e^{-p_0(z)}|f(z)|^2\,dx\,dy < +\infty.$$

In the space $L_a^2[\Omega, p_0]$, we define the inner product by

$$(g(z), f(z)) = \int\int_{\Omega} e^{-p_0(z)} g(z)\overline{f(z)}\,dx\,dy,$$

where $f(z), g(z) \in L_a^2[\Omega, p_0]$.

THEOREM 2. *Under the conditions of Theorem 1, the sequence* $\{z^{\tau_1}, z^{\tau_2}, \ldots\}$ *is complete in* $L_a^2[\Omega, p_0]$.

The proof is almost the same as that of Theorem 1. We only need to note the following points:

In the estimate of the upper bound of $|G(s)|$, we now have for $s \in Q_\gamma^\delta$,

$$G(s) = \int\int_{\Omega'} e^{-p_0(e^\xi)}\overline{f(e^\xi)}|e^\xi|^2 I(s-\xi)\, d\xi_1\, d\xi_2,$$

and for $\mathrm{Re}(s) = u > 0,\ s \in Q_\gamma^\delta$,

$$|G(s)| \leqslant C_1^{t_n} \frac{\int\int_\Omega e^{-p_0(z)}|f(z)||z|^{t_n}\, dx\, dy}{|e^s|^{t_n} \sin(\mu\pi)}$$

and

$$\begin{aligned}
&\int\int_\Omega e^{-p_0(z)}|f(z)||z|^{t_n}\, dx\, dy \\
&\quad = \int\int_\Omega e^{-1/2 p_0(z)}|f(z)|e^{-1/2 p_0(z)}|z|^{t_n}\, dx\, dy \\
&\quad \leqslant C_1 \left[\int\int_\Omega e^{-p_0(z)}|z|^{2t_n}\, dx\, dy\right]^{1/2} \\
&\quad \leqslant C_2^{t_n} \left[\int_{r_0}^\infty re^{-p_0(r)} r^{2t_n}\, dr\right]^{1/2} \\
&\quad \leqslant C_2^{t_n} \left[\sup_{r \geqslant 0} (re^{-1/2 p_0(r)}\right]^{1/2} \left[\int_{r_0}^\infty e^{-1/2 p_0(r)} r^{2t_n}\, dr\right]^{1/2} \\
&\quad \leqslant C_3^{t_n} \left[\int_{r_0}^\infty e^{-1/2 p_0(r)} r^{2t_n}\, dr\right]^{1/2} \\
&\quad \leqslant C_3^{t_n} \left[\int_{r_0}^\infty e^{-1/2 p(r)} r^{2t_n}\, dr\right]^{1/2},
\end{aligned}$$

where C_1, C_2 and C_3 are constants independent of t_n and n. Then, as in the proof of Theorem 1, we can prove that $G(s) \equiv 0$ for $s \in Q_\gamma^\delta$.

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